RF

By Zack Lau, W1VT

Proving the Conjugate Matching Power Theorem

This theorem has provoked quite a bit of controversy in amateur circles in recent years. I think it might be useful for advanced amateurs to see a mathematical proof. A good proof often adds to one's understanding of concepts.

Let's start by defining our system: The power source is represented by a Thevenin equivalent, which is a source voltage and a series source impedance, represented by *V*s and $R \pm jX$, respectively (see Figure 1). Radio books use *j* to represent the imaginary operator because the usual imaginary operator, "*i*," is already used to represent current. Math books use "*i*" to represent the imaginary operator; that is, $i^2 = -1$. All of these are constants, invariant with time. Now we seek to determine the optimum load impedance, Z_l (that subscript is a lower-case "L"—*Ed*), which is composed of a resistor R_l and series impedance X_l. By optimum, we mean that the power lost in R_l is maximized.

Since we have a series circuit, the power lost, P_l, is most easily calculated in terms of $|i|^2 \bullet R_l$. Since R_l is defined as a resistor, we just need to know the magnitude of the current through it. Thus, we can calculate $|i|^2$ as $(\bar{i}) \bullet (i)$, where $\bar{i}$ is the complex conjugate of *i*.

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$$i = \frac{V_s}{R + jX + jX_l + R_l} \qquad \text{(Eq 1)}$$

$$\bar{i} = \frac{V_s}{R - jX - jX_l + R_l} \qquad \text{(Eq 2)}$$

$$\begin{aligned} i \bullet \bar{i} &= \frac{V_s^2}{\left(\left((R+R_l)+j(X+X_l)\right)\left((R+R_l)-j(X+X_l)\right)\right)} \\ &= \frac{V_s^2}{(R+R_l)^2-\left(j^2\right)(X+X_l)^2} \\ &= \frac{V_s^2}{(R+R_l)^2+(X+X_l)^2} \end{aligned} \qquad \text{(Eq 3)}$$

Taking the complex conjugate results in the sum of squares and cancellation of the imaginary cross product.

Thus, the power lost, P_l, is

$$P_l = \frac{V_s^2 \bullet R_l}{\left((R+R_l)^2+(X+X_l)^2\right)} \qquad \text{(Eq 4)}$$

Now, we wish to see how to maximize P_l by optimizing Z_l. Since Z_l is actually $R_l + jX_l$, we need to optimize P_l with respect to R_l and X_l. Calculus was invented for just this purpose. For example, if you were given an equation of the distance traveled from a point by a car, you could determine its maximum distance by just looking at the times the speed drops to zero, as well as looking at its starting and stopping times. This is quite useful—an infinite number of values is reduced to just a few points on a graph that can be easily calculated. Similarly, we can use calculus to determine the key points of complex electronic equations, to better understand how circuits work.

To do this, we differentiate P_l with respect to R_l and X_l. Remembering that if the equation is of the form

$$\frac{f(x)}{g(x)} \qquad \text{(Eq 5)}$$

the derivative is

$$\frac{f'(x)g(x)-g'(x)f(x)}{\left(g(x)\right)^2} \qquad \text{(Eq 6)}$$

One of the tricks to doing math is to simplify the equations as much as possible. I like to look for terms that cancel or become zero. Thus, it makes a lot of sense to do the differentiation with respect to X_l first. This way, $f(x)$ is a constant, which means that $f'(x) = 0$.

$$f(x) = V_s^2 \bullet R_l \qquad \text{(Eq 7)}$$

$$g(x) = \left(R+R_l\right)^2 + \left(X+X_l\right)^2 \qquad \text{(Eq 8)}$$

$$f'(x) = 0 \qquad \text{(Eq 9)}$$

$$g'(x) = 2 \bullet \left(X+X_l\right) \qquad \text{(Eq 10)}$$

Thus,

$$\frac{dP_l}{dX_l} = \frac{\left(0 \bullet g(x) - 2 \bullet \left(X+X_l\right)\left(V_s^2 \bullet R_l\right)\right)}{\left(\left(R+R_l\right)^2 + \left(X+X_l\right)^2\right)^2} \qquad \text{(Eq 11)}$$

It isn't necessary to expand $g(x)$—we are just going to cross it out.

$$\text{Setting } \frac{dP_l}{dX_l} = 0, \quad V_s^2 \bullet R_l \bullet 2\left(X+X_l\right) = 0$$

$$X_l = -X \qquad \text{(Eq 12)}$$

Keep in mind that $(R + R_l)^2$ cannot equal zero, or we would have division by zero. Very bad mistakes can occur when you divide by zero. Thus, R_l is not equal to $-R$.

Alternatively, we can look at the power equation with R_l set to an arbitrary constant. Since the numerator is fixed, we can maximize the power by minimizing the denominator. This occurs when $X_l + X$ is set equal to zero. Thus, $X_l = -X$. This approach is not as rigorous, but a knowledge of calculus isn't required. We can use $X + X_l = 0$ to simplify the equation,

$$P_l = \frac{V_s^2 \bullet R_l}{\left(R+R_l\right)^2 + \left(X+X_l\right)^2} \qquad \text{(Eq 13)}$$

$$P_l = \frac{V_s^2 \bullet R_l}{\left(R+R_l\right)^2} \qquad \text{(Eq 14)}$$

We can now take a partial derivation with respect to R_l.

$$f(x) = V_s^2 \bullet R_l \qquad \text{(Eq 15)}$$

$$g(x) = \left(R+R_l\right)^2 \qquad \text{(Eq 16)}$$

$$f'(x) = V_s^2 \qquad \text{(Eq 17)}$$

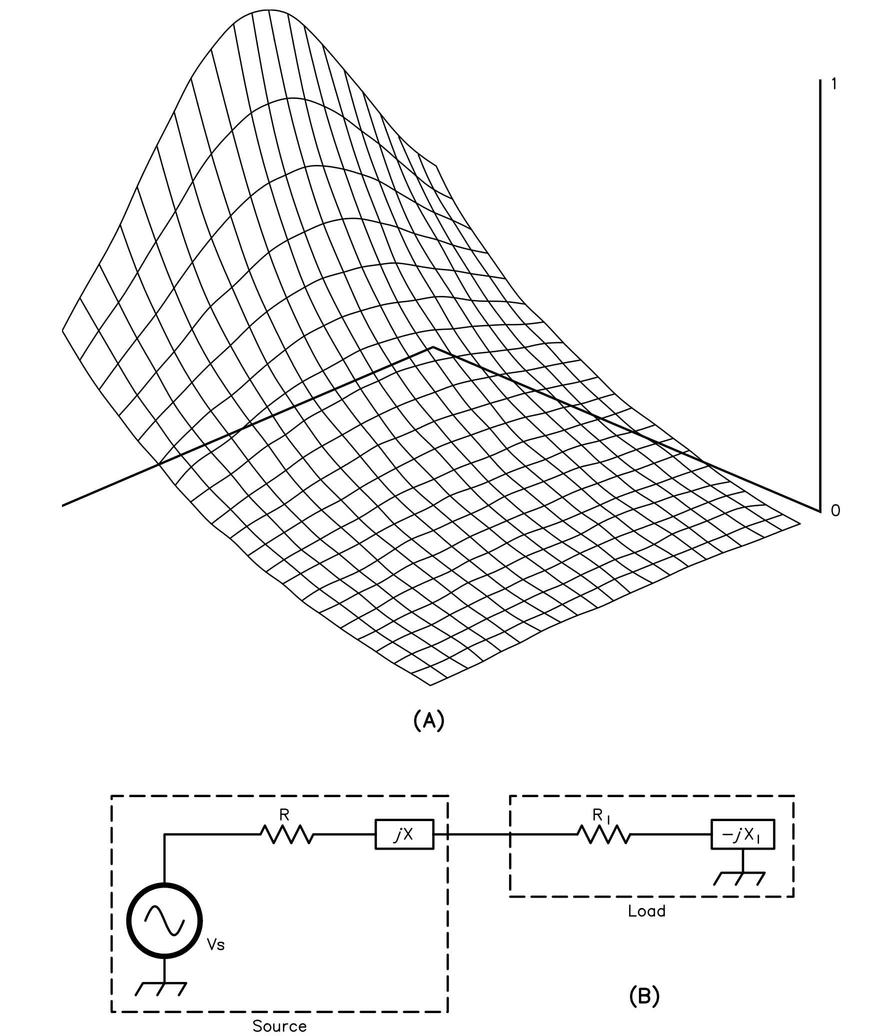

Fig 1—At A, a graph of power lost (P_l) as a function of R and X. See Table 1 for the conditions associated with this graph. At B, a diagram of the source and load circuits.

Table 1—Conditions associated with Fig 1

$$R_l = 1$$

$$X_l = 1$$

$$N = 20$$

$$n = 0 \cdots N$$

$$m = 0 \cdots N$$

$$R_n = 0.1n$$

$$X_m = 0.1m$$

$$f(R,X) = \frac{R_l}{\left(R+R_l\right)^2 - \left(X+X_l\right)^2}$$

$$PL_{(m,n)} = f\left(R_n, X_m\right)$$

$$g'(x) = 2(R + R_l) \quad \text{(Eq 18)}$$

$$\frac{dP_l}{dR_l} = \frac{\left(V_s^2 \bullet (R+R_l)^2 - 2(R+R_l)V_s^2 \bullet R_l\right)}{\left((R+R_l)^2\right)^2} \quad \text{(Eq 19)}$$

Setting $dP_l / dR_l = 0$ and noting that $R + R_l$ cannot equal zero (lest we divide by zero):

$$V_s^2 \bullet (R+R_l)^2 = 2(R+R_l)V_s^2 \bullet R_l \quad \text{(Eq 20)}$$

$$R^2 + 2RR_l + R_l^2 - 2RR_l - 2R_l^2 = 0 \quad \text{(Eq 21)}$$

$$R^2 - R_l^2 = 0 \quad \text{(Eq 22)}$$

Factoring, $(R + R_l)(R - R_l) = 0$

Remember that we previously noted that R is not equal to $-R_l$, to avoid division by zero. Thus, the only solution is $R_l = R$.

Since the real parts of the source and load impedance are equal, the source and load lose the same amount of power, setting the system efficiency at 50%.

Now to be very rigorous, you could do second differentiation, to determine whether the point we found is a maximum, a minimum or an inflection point. Alternatively, we know from practical experience or calculations that either endpoint—$Z_l = 0$ or $Z_l = \infty$ results in zero power transfer to the load. Also, power increases as the real part increases from zero or decreases from infinity. Thus, it is quite obvious from practical experience that the single point obtained is the desired maximum. Even if you do perform the extra math, looking at equations from this viewpoint is always a good idea. Remember that the equations are intended to simulate reality.

Thus, we have proven that if the source impedance is $R + jX$, the load impedance that results in maximum power transfer is $R - jX$. Thus, the optimum load impedance is the complex conjugate of the source impedance.

So why the controversy? I think it results from the misapplication of the theorem to modern transmitters. Most amateurs want to maximize their output power, so they look to the conjugate-matching theorem for guidance. However, modern transmitters are designed to work into specific load impedances, usually 50 Ω, rather than to be conjugately matched. While you may get more power with a conjugate match, you may overstress the final amplifier parts or create excessive distortion. Thus, even though I use a meter that automatically subtracts the reverse power from the forward power to give actual power measurements, I still adjust my Transmatch for minimum SWR, rather than maximum actual power.[1] An SWR meter measures how closely you match a desired impedance with a single number, which makes it quite convenient as a tuning indicator.

Many thanks to Kevin Schmidt, W9CF, who looked at an early draft of this manuscript and provided useful comments.

Why are SWR Meters affected by Power Level?

Contrary to what some believe, the impedance of the source has little to do with what an SWR meter reads. An SWR meter reads reflections that return from the load, not the source. A reflection from the source would be indistinguishable from "forward" power generated by the source.

The greatest source of erroneous SWR readings is nonlinearity of diode detectors. Consider an oversimplified model of an ideal –32-dB directional coupler with diodes having an exact 0.3 V conduction voltage. (See Fig 2.) The –32-dB coupling factor means that the forward power detector samples 32 dB less signal than is applied to the coupler. For instance, a 100-W signal will result in 50 dBm – 32dB = +18 dBm of signal, or 63 mW applied to the forward power detector. On the other hand, the 0.3-V voltage drop means that $(0.3\ \text{V})^2/(2 \times 50\ \Omega)$ must be generated before the diode will conduct. This is 0.9 mW or –0.5 dBm. Thus, the reverse power can be has high as 0.9 mW before *any* reflected power will be indicated. At that level, the actual SWR would be

[1]"The Tandem Match—An Accurate Directional Wattmeter, John Grebenkemper, KI6WX, January 1987 *QST* and recent ARRL Handbooks p.22.34 to 22.40 of the *2000 Handbook*.

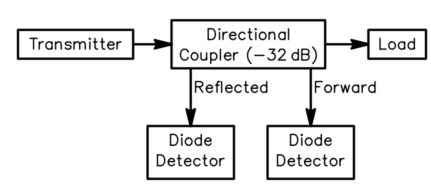

Fig 2—Block diagram of a typical SWR meter.

$$SWR = \left(\frac{1 + \left(\frac{P_R}{P_F}\right)^{0.5}}{1 - \left(\frac{P_R}{P_F}\right)^{0.5}} \right) = \left(\frac{1 + \left(\frac{0.9\ \text{mW}}{63\ \text{mW}}\right)^{0.5}}{1 - \left(\frac{0.9\ \text{mW}}{63\ \text{mW}}\right)^{0.5}} \right) = 1.27 \quad \text{(Eq 23)}$$

where P_F is the forward power and P_R is the reflected power. Thus, the meter reads a 1.0 SWR when the SWR may actually be as high as 1.27.

If the power is boosted to 1500 W, the forward-power detector sees 61.8 dBm – 32 dB, or 29.8 dBm, or 955 mW.

The reverse power detector sees 0.9 mW × 1500 W / 100 W = 13.5 mW. This is 0.821 V (RMS) across a 50-Ω load, or 1.16 V (peak). Subtracting the 0.3 V, one gets an apparent reflected power of 7.4 mW.

Similarly, the apparent forward power is 897 mW, due to the diode drop.

$$SWR = \left(\frac{1 + \left(\frac{7.4\ \text{mW}}{897\ \text{mW}}\right)^{0.5}}{1 - \left(\frac{7.4\ \text{mW}}{897\ \text{mW}}\right)^{0.5}} \right) = 1.20 \quad \text{(Eq 24)}$$

Thus, the SWR may appear to jump from 1.0 to 1.20 when an amplifier boosts the power from 100 W to 1500 W.

The situation becomes worse when running QRP. At the 5-W level, the forward power available from the directional coupler is just 37 – 32 = +5 dBm, or 3 mW. Since the reverse power meter reads zero for signals up to a threshold of 0.9 mW, the SWR could get as high as

$$\frac{1 + \left(\frac{0.9}{3}\right)^{0.5}}{1 - \left(\frac{0.9}{3}\right)^{0.5}} = 3.4 \quad \text{(Eq 25)}$$

Table 2—Summary of SWR calculations

Transmit Power (W)	*Forward Power at Detector*	*Reverse Power at Detector*	*Apparent Forward power*	*Apparent Reflected Power*	*Apparent SWR*
100	63 mW	0.9 mW	49 mW	0 mW	1.00
1500	955 mW	13.5 mW	897 mW	7.4 mW	1.20

Thus, the SWR could be 3.4:1 before the reflected power begins to register, even if the forward scale is properly calibrated. This is summarized in Table 2.

A solution is to draw separate SWR scales for 5- and 100-W levels that compensate for the diode nonlinearity. Alternately, one can dispense with the SWR scale and mentally compute the degree of mismatch as the ratio of the forward and reverse powers. Bird Corporation suggests this approach with their analog power meters.

Another approach is to use a better detector—one that corrects for the nonlinearity of the diode detector. Roy Lewallen, W7EL, published an excellent example of this approach in "A Simple and Accurate QRP Directional Wattmeter," (*QST*, Feb 1990, pp 19-23). Roy discovered that not only is the diode drop a problem, but that best results require calibration with ac, rather than dc, reference signals. His article also gives a more precise model of diodes that allows more-accurate calculations of circuit performance.

Trepanning Large Holes

At the 1999 Microwave Update, Ed Krome, K9EK, talked about one of the most hazardous modern Amateur Radio activities: fly cutting a large hole in a sheet of metal. A sharp metal bit is held in a fixture that allows it to rotate in a large circle, cutting a grove in piece of metal firmly attached to a drill press. This can be quite dangerous if the bit gets stuck—the metal can easily be sent flying through the air. Often, the cutting operation has enough vibration to loosen any clamps, allowing the metal to fly free. A poor solution is to carefully smooth the edges and round the corners of the sheet metal. This will prevent the projectile from acting like a sharp knife.

A better solution is to use a milling machine and a rotary table to trepan the holes. Instead of whirling around a big cutter, the sheet metal is slowly rotated with the rotary table. The milling machine just cuts a little 1/16 or 7/64 inch hole as the plate is rotated. Granted, a little four-inch Sherline rotary table won't allow exact duplication of a big 432 MHz plate line, but will often do a fine job at higher frequencies. It may also be practical to make precision plate collets that can be screwed onto low-frequency plate lines. I suspect that little four-inch plates are much easier to work with, particularly if you are attempting to preheat the work with a hot plate. The milling machine is also useful for empirically optimizing your work. It isn't too difficult to mill off the used solder and enlarge a hole, if you want to try a slightly larger hole.

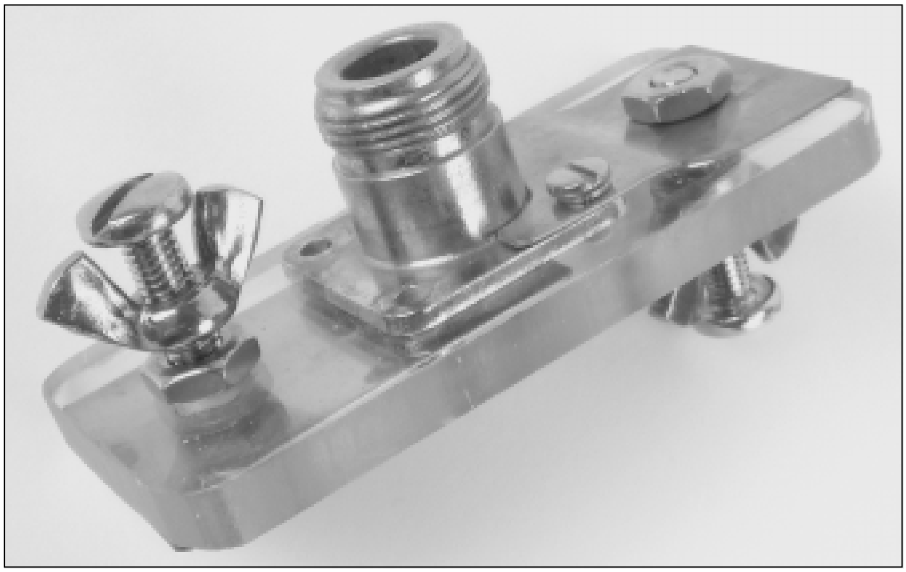

Fig 2—W3IRZ's technique to capture wing nuts at connections.

More on Center Insulator for Dipoles

Zack—That's an okay idea on the wing nuts for binding posts, but here's one better: For field use, you should prevent the loss of the wing nuts by reversing the screw 180°. By doing this, the wing nut will contact the screw head when it unscrews about 3/8 of an inch. The screw is held onto the material (plastic in this case) with a nut on both sides. I picked up a box of #8-32 wing nuts years ago and have been using them in this manner for some time.—*Mike Branca, W3IRZ, Conyers, Georgia;* **w3irz@att.net**

Next Issue in QEX/Communications Quarterly

Bar-Giora Goldberg gives us a review of popular frequency-synthesis techniques and his vision for the future of that field. He discusses tradeoffs among power consumption, spectral purity, cost and complexity. Interesting information regarding loop noise shaping and all-digital fractional-N methods is presented. Giora also concentrates on characterizing signals as narrow-band noise and points out a prime goal of frequency synthesis: cleaning up the signal.

As we've seen recently, interest in homebrewing high-dynamic-range receivers has persuaded some designers to rethink the use of broadband front ends. Bill Sabin, WØIYH, brings us his design of a bank of narrow band-pass filters that may be used ahead of receivers to limit input bandwidth, thus assuaging second-order IMD and other problems. CAD is used extensively in design and analysis of the filters. To achieve close linearity, Bill paid careful attention to the inductor-core material and flux density.

Charles Kitchin, N1TEV, updates us on some new super-regenerative receiver techniques. He draws on his experience with these circuits over the last few years to bat down some false—but common—notions about them. Charles details traditional drawbacks of the "super-regen" and tells how he and others have developed ways to avoid them. Designs are discussed for VHF and NBFM.

R. P. Haviland, W4MB's series on quad antennas continues. Part 3 covers the care and feeding of multi-element designs.